(1)

Banach Colmez Spaces

Fontaine "Presque $\mathbb{C}_p$-représentations"
Colmez "Espaces de Banach de dimension finie"
Le Bras "Espaces de Banach-Colmez et faisceaux cohérents sur la Courbe de F.F."

$C/\mathbb{Q}_p$ alg. closed

Perf_C = C-perfectoid spaces + pro-étale topology
$= \mathrm{Perf}_{C^\flat}$

$\underline{\mathbb{Q}_p}$-Mod = Pro-étale sheaves of $\underline{\mathbb{Q}_p}$-modules on Perf_C

finite dim.

$$* \, \mathrm{Vect}_{\mathbb{Q}_p} \hookrightarrow \underline{\mathbb{Q}_p}\text{-Mod}$$
$$V \longmapsto \underline{V}$$

$$* \, \mathrm{Vect}_C \hookrightarrow \underline{\mathbb{Q}_p}\text{-Mod}$$
$$W \longmapsto W \otimes_C \mathbb{G}_a$$

More general fact: Normal C-rigid spaces $\hookrightarrow \widetilde{\mathrm{Perf}_C}$ via $X \mapsto \mathrm{Hom}(-, X)$

Def: $\mathcal{BC}$ = smallest abelian category stable under extensions of $\underline{\mathbb{Q}_p}$-Mod containing $\mathrm{Vect}_{\mathbb{Q}}$ and Vect_{C}.

Ex: * $\mathbb{B}^+_{dR}/\mathrm{Fil}^r$ since $\mathrm{Fil}^i/\mathrm{Fil}^{i+1} \simeq \mathbb{G}_a$

$\mathbb{B}^+_{dR}(R,R^+)$ = ker θ-adic completion of $W(R^{\flat,\circ})[\frac{1}{p}]$

$\theta : W(R^{\flat,\circ}) \twoheadrightarrow R^\circ$

* $\mathbb{B}^{\varphi=p^d}$ - Fundamental exact sequence of p-adic Hodge theory

$(R,R^+) \longmapsto B(R^\flat,R^{\flat,+})^{\varphi=p^d}$

$$0 \longrightarrow \mathbb{B}^{\varphi=p^{d-1}} \xrightarrow{\times t} \mathbb{B}^{\varphi=p^d} \xrightarrow{\theta} \mathbb{G}_a \longrightarrow 0$$

$+ \ \mathbb{B}^{\varphi=Id} = \underline{\mathbb{Q}_p} \Rightarrow$ by induction $\mathbb{B}^{\varphi=p^d} \in \mathcal{BC}$.

②

$X =$ curve associated to $C^\flat$ + $\infty \in |X|$, $k(\infty) = C$

$$\tau : \widetilde{X}_{pro\text{-}\acute{e}t} \longrightarrow \mathrm{Spa}(C^\flat)^{\sim}_{pro\text{-}\acute{e}t} = \mathrm{Spa}(C)^{\sim}_{pro\text{-}\acute{e}t}$$

$$(\tau_* \mathcal{F})(U) = H^0(X_U, \mathcal{F}_{|X_U})$$

<u>Torsion structure on Coh_X</u>: $\left(\mathrm{Coh}_X^{\geq 0}, \mathrm{Coh}_X^{<0}\right)$

Coherent sheaves wt. HN slopes ≥ 0 (in particular torsion coherent sheaves wt. slope $+\infty$)

v.b. with slopes <0

$\rightsquigarrow$ t-structure on $\mathcal{D} = \mathbb{D}^b_{\mathrm{Coh}}(\mathcal{O}_X)$

$$\mathcal{D}^{\geq 0} = \left\{ \mathcal{E} \in \mathcal{D} \;\middle/\; \begin{array}{l} \mathcal{H}^i(\mathcal{E}) = 0 \text{ if } i < -1 \\ \mathcal{H}^{-1}(\mathcal{E}) \in \mathrm{Coh}_X^{<0} \end{array} \right\}$$

$$\mathcal{D}^{\leq 0} = \left\{ \mathcal{E} \in \mathcal{D} \;\middle/\; \begin{array}{l} \mathcal{H}^i(\mathcal{E}) = 0 \text{ if } i > 0 \\ \mathcal{H}^0(\mathcal{E}) \in \mathrm{Coh}_X^{\geq 0} \end{array} \right\}$$

$$\widehat{\mathrm{Coh}}_X = \text{Heart}$$

$$\underbrace{\phantom{\widehat{\mathrm{Coh}}_X}}_{\text{abelian category}} = \left\{ \mathcal{E} \in \mathbb{D}^{[-1,0]}_{\mathrm{Coh}}(\mathcal{O}_X) \;\middle|\; \mathcal{H}^{-1}(\mathcal{E}) \in \mathrm{Coh}_X^{<0} \text{ and } \mathcal{H}^0(\mathcal{E}) \in \mathrm{Coh}_X^{\geq 0} \right\}$$

Th (Arthur-César Le Bras):

* $\underbrace{R\tau_*}_{\text{relative cohomology functor}} : \widehat{\mathrm{Coh}}_X \xrightarrow{\sim} \mathcal{BC}$

* $\mathbb{D}^b(\widehat{\mathrm{Coh}}_X) \xrightarrow{\sim} \mathbb{D}^b(\mathrm{Coh}_X)$

  ↳ Beilinson functor

* $\deg, \mathrm{rk} : K_0(\mathrm{Coh}_X) \begin{smallmatrix} \xrightarrow{\deg} \mathbb{Z} \\ \xrightarrow[\mathrm{rk}]{} \mathbb{N} \end{smallmatrix}$

$$\begin{aligned} &\| \\ K_0&(\mathbb{D}^b_{\mathrm{Coh}}(\mathcal{O}_X)) \\ &\| \\ K_0&(\widehat{\mathrm{Coh}}_X) \end{aligned}$$

Invariance of the K_0 by change of the t-structure

$\rightsquigarrow \deg, \mathrm{rk} : \widehat{\mathrm{Coh}}_X \to \mathbb{Z}$ additive functions

$$\deg \mathcal{E}^\cdot = \deg \mathcal{H}^0(\mathcal{E}^\cdot) - \deg \mathcal{H}^{-1}(\mathcal{E}^\cdot)$$

$$\mathrm{rk}\, \mathcal{E}^\cdot = \mathrm{rk}\, \mathcal{H}^0(\mathcal{E}^\cdot) - \mathrm{rk}\, \mathcal{H}^{-1}(\mathcal{E}^\cdot)$$

③

define $\begin{cases} \widehat{\deg} := -\mathrm{rk} : \widehat{\mathrm{Coh}}_X \to \mathbb{Z} \\ \widehat{\mathrm{rk}} := \deg : \mathrm{Coh}_X \to \mathbb{N} \end{cases}$

$\hat{\mu} = \dfrac{\widehat{\deg}}{\widehat{\mathrm{rk}}} \rightsquigarrow$ H.N. filtration for $\hat{\mu}$

(slope $-\infty$ objects $= \mathrm{Bun}_X^{ss,0} = \mathrm{Vect}_{\mathbb{Q}_p}$)

Then $*$ if $\lambda > 0$ slope λ s.s. objects in $\widehat{\mathrm{Coh}}_X$ are finite direct sums of $\mathcal{O}_X(-\frac{1}{\lambda})[1]$

$*$ if $\lambda < 0$ " " $\mathcal{O}_X(-1/\lambda)$

$*$ if $\lambda = 0$ " " $\mathcal{F}$ with $\mathcal{F} \in \mathrm{Coh}_X^{tor}$

$*$ if $\lambda = -\infty$ " " $\mathcal{O}_X$

The HN filtration of an object of $\widehat{\mathrm{Coh}}_X$ is split.

$*$ Each $\mathcal{F} \in \mathcal{BC}$ is represented by a diamond!!

Ex: $\lambda = \frac{d}{h} \in]0,1] \cap \mathbb{Q}$

$\mathcal{G}$ = formal p-divisible group of dimension d and height h $/\overline{\mathbb{F}}_p$

$$\tilde{\mathcal{G}} := \varprojlim_{\times p} \mathcal{G} = \varprojlim_{\mathrm{Frob}} \mathcal{G} \simeq \mathrm{Spf}\left(\overline{\mathbb{F}}_p[\![x_0^{1/p^\infty}, \dots, x_{d-1}^{1/p^\infty}]\!]\right)$$

$\underbrace{\phantom{\tilde{\mathcal{G}}}}$ formal $\mathbb{Q}_p$-vector space = universal cover of $\mathcal{G}$

Then $$\tau_* \mathcal{O}(\lambda) \simeq \tilde{\mathcal{G}}_{C^\flat} \simeq \underbrace{\mathring{\mathbb{B}}_C^{d, 1/p^\infty}}_{\text{perfectoid open ball } /C} \in \mathrm{Perf}_C$$

<u>Concretely:</u> (R, R^+) = perfectoid $\overline{\mathbb{F}}_q$-algebra

$$(R^{\circ\circ})^d \xrightarrow{\sim} \mathbb{B}(R, R^+)^{\varphi^h = p^d}$$

$$(x_0, \dots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{b \in \mathbb{Z}} [x_i^{q^{-b}}]\, p^{bd+i}$$

$\Longrightarrow$ For $\lambda \in [0,1] \cap \mathbb{Q}$, $\tau_* \mathcal{O}(\lambda)$ represented by a perfectoid space.

False if $\lambda > 1$, not representable by a perfectoid space.

Let $d > 1$, $d \in \mathbb{N}$, one has an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{d-1} \xrightarrow{u} \mathcal{O}_X(1)^d \xrightarrow{v} \mathcal{O}_X(d) \longrightarrow 0$$

$\rightsquigarrow$ choose $t_1, \dots, t_d \in H^0(\mathcal{O}(1)) \setminus \{0\}$ s.t. $V(t_i) \cap V(t_j) = \phi$ if $i \neq j$

one point

Let $\hat{t}_i = \prod_{j \neq i} t_j \in H^0(\mathcal{O}(d-1))$

Koszul type resolution

$$v(\lambda_1, \dots, \lambda_d) = \sum_{i=1}^{d} \lambda_i \hat{t}_i$$

Let $V = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{Q}_p^d \,/\, \sum_i \lambda_i = 0\}$

$$u : V \otimes_{\mathbb{Q}_p} \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^d$$

$$(\lambda_1, \dots, \lambda_d) \longmapsto \sum_{i=1}^{d} \lambda_i t_i$$

Can apply this to X_h, $h \geq 1$

$$\begin{array}{l} X_h = X \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^h} \\ \downarrow \pi_h \\ X \end{array} \Bigg) \ \mathbb{Z}/h\mathbb{Z}\text{-cycle covering.}$$

$$0 \to \mathcal{O}_{X_h}^{d-1} \to \mathcal{O}_{X_h}(1)^d \longrightarrow \mathcal{O}_{X_h}(d) \to 0$$

$\rightsquigarrow$ apply π_{h*}

$$\rightsquigarrow 0 \to \mathcal{O}_X^{h(d-1)} \to \mathcal{O}_X\left(\tfrac{1}{h}\right)^d \longrightarrow \mathcal{O}_X\left(\tfrac{d}{h}\right) \to 0$$

<u>Apply $R\tau_*$</u> to obtain an exact sequence of BC's

$$0 \to \underline{\mathbb{Q}_p}^{h(d-1)} \longrightarrow \underbrace{\tau_* \mathcal{O}_X\left(\tfrac{1}{h}\right)^d}_{\text{perfectoid space}} \xrightarrow{\text{pro-étale cover}} \tau_* \mathcal{O}_X\left(\tfrac{d}{h}\right) \longrightarrow 0$$

* The case of $\mathcal{O}_X(-\lambda)[1]$ with $\lambda > 0$

i.e. $R^1\tau_* \mathcal{O}_X(-\lambda)$ = relative H^1 of $\mathcal{O}_X(-\lambda)$

$\lambda = d > 0$, $d \in \mathbb{N}$

$t \in H^0(X, \mathcal{O}_X(1)) \setminus \{0\}$, $V^+(t) = \infty$

$$0 \to \mathcal{O}_X(-d) \xrightarrow{\times t^d} \mathcal{O}_X \to i_{\infty *} B^+_{dR} / \mathrm{Fil}^d \to 0$$

$$\Rightarrow R^1 \tau_* \mathcal{O}(-d) \simeq \mathbb{B}^+_{dR} / \mathrm{Fil}^d + \underline{\mathbb{Q}_p}$$

__Ex.__ $R^1 \tau_* \mathcal{O}(-1) = \mathbb{G}_a / \underline{\mathbb{Q}_p}$

__More generally:__ $\lambda = \frac{d}{h} > 0$

$$R^1 \tau_* \mathcal{O}(-\lambda) \simeq \mathbb{B}^+_{dR} / \mathrm{Fil}^d + \underline{\mathbb{Q}_{p^h}}$$ diamond.

__Ex.__

diamond

$$\mathbb{G}_a^\diamond \twoheadrightarrow \mathbb{G}_a / \underline{\mathbb{Q}_p}$$

pro-étale cover